JOURNAL OF GUIDANCE, CONTROL, AND DYNAMICS
Vol. 31, No. 1, January–February 2008

Fuel-Equivalent Relative Orbit Element Space

Jean-Francois Hamel* and Jean de Lafontaine†
Universite de Sherbrooke, Sherbrooke, Quebec J1K 2R1, Canada

DOI: 10.2514/1.32034

This paper presents a new tool to analytically perform the guidance for reconfiguration of formation flying spacecraft. The technique consists in mapping the relative orbit elements into a fuel-equivalent space where similar displacements correspond to an equivalent fuel consumption. The minimal-fuel maneuver problem is consequently translated into a simple geometric problem in the fuel-equivalent space. The theory is applied to two well-known formations: the J_2-invariant formation and the projected circular formation. The use of the fuel-equivalent space leads to very simple solutions for the most fuel-efficient way to attain both formations.

I. Introduction

THERE has been undoubtedly a paradigm shift in the last years toward the use of spacecraft formation flying. Formation flying replaces large and expensive spacecraft by several smaller spacecraft that can perform the same mission with an increased reconfigurability and robustness to failures. However, formation flying increases the complexity of some systems, mainly of the guidance, navigation, and control functions, which rapidly grow in complexity with the number of spacecraft in the formation.

This conflicts, however, with the increasing need for autonomy to decrease the cost of ground support. Ground support operations are still a non-negligible part of the cost of a mission, especially for small scientific missions with small budgets. This naturally leads to a need for more autonomous guidance, navigation, and control algorithms that can perform autonomous decisions and tradeoffs that would otherwise be performed by the ground segment. This also becomes very challenging for formations with a large number of spacecraft.

This paper therefore concentrates on the development of a new tool to autonomously perform formation flying guidance. The purpose of the guidance system is to provide a reference trajectory to reach a specific formation. This trajectory can optimize the duration of the maneuver, optimize the fuel cost of the maneuver, minimize the risk of collision, or do all three at the same time.

The most common approach in formation flying guidance is the use of computationally expensive techniques. Such examples are the use of linear programming [1,2], multi-agent optimization techniques [3], particle swarm optimization [4], genetic algorithms [5], or optimal control theory [6,7]. These kinds of techniques have a lot of freedom in the selection of the quantity to optimize and the constraints to impose. They all provide ways to compute the best maneuver to reach a desired formation. However, these methods require an initially unknown (and most likely large) number of iterations, and convergence is not always guaranteed. Obviously, this precludes any on-board implementation of this type of algorithm.

On the other hand, analytical solutions to the optimal reconfiguration problem can be found under certain conditions. Indeed, unperturbed circular reference orbits lead to simple analytical expressions and easily expressed configurations [8]. Mishne [9] almost analytically solves the optimal control problem for circular orbits for power-limited thrusters (only a small amount of numerical optimization remains). Furthermore, Vaddi et al. [10] developed an analytical and simple solution to the circular formation establishment and reconfiguration using impulsive thrusters about a circular reference orbit. On the other hand, Gurfil [11] proposes an analytical and optimal way of reaching bounded relative motion for any Keplerian orbits with only one impulse through the application of an energy-matching constraint. However, even though this impulse guarantees orbit-periodic relative motion, it is not made to aim for a specific configuration.

It is the intent of this paper to propose a new tool that yields an analytical solution to the fuel-optimal reconfiguration problem for any type of orbit for any geometrically simple formation. To do so, spacecraft relative positions and desired formations are mapped into a fuel-equivalent space in which equivalent distances on all axes relate to identical fuel consumption. This mapping is a way to rapidly compute the most fuel-efficient way to reach a formation by taking the shortest path in the fuel-equivalent space, reducing the problem to a simple geometric problem. It avoids the need to perform a systematic search as is traditionally done [2].

Section II first reviews the impulsive feedback controller [12,13] upon which the fuel-equivalent space theory is built. Then, Sec. III defines the fuel-equivalent space. Finally, Secs. IV and V provide two examples of how this theory can be applied to compute the most fuel-efficient maneuvers for two well-known formation flying cases: the J_2-invariant relative orbits and the projected circular formation (PCF).

II. Impulsive Feedback Controller

The impulsive feedback controller [12,13] was proposed as a way to perform orbit element corrections while minimizing the impact on the other orbit elements. It is based on the Gauss variational equations and can perform any arbitrary small orbit correction with only three impulses. If only one or two elements are to be corrected, the controller provides essentially optimal results in terms of fuel. If all six elements are to be corrected, the controller proposes maneuvers that are only a few percents larger than the optimal multi-impulse solution. However, the most important advantage of this technique is that the impulses and their locations can be computed analytically with very simple expressions, leading very quickly to a good approximation of the fuel cost of a maneuver, even if the spacecraft does not make use of impulsive thrusters.

The controller performs the corrections $\Delta \boldsymbol{e} = [\Delta a \;\; \Delta e \;\; \Delta i \;\; \Delta\Omega \;\; \Delta\omega \;\; \Delta M]^T$ on all six orbital elements (i.e., the semimajor axis a, the eccentricity $\boldsymbol{e}$, the inclination i, the right ascension of the ascending node Ω, the argument of periapsis ω, and the mean anomaly M) with only three impulses, Δv_p, Δv_a, and Δv_h, respectively at the periapsis, at the apoapsis, and at a critical true latitude angle θ_c.

The first impulse, Δv_h, performs both inclination and ascending node corrections in one single normal impulse. Obviously, it is more efficient to correct the inclination when the spacecraft crosses the

---

Presented at the AAS/AIAA Astrodynamics Specialist Conference, Mackinac Island, Michigan, 19–23 August 2007; received 8 May 2007; revision received 3 August 2007; accepted for publication 11 August 2007.

*Graduate Student, Department of Electrical Engineering. Student Member AIAA.

†Professor, Department of Electrical Engineering. Member AIAA.

equator and to correct the ascending node near the poles, but it is more fuel efficient to correct both elements with one single impulse if both have to be corrected. This normal impulse is to take place at the critical true latitude angle θ_c:

$$\theta_c = \arctan\frac{\Delta\Omega \sin i}{\Delta i} \tag{1}$$

and its magnitude is

$$\Delta v_h = \sqrt{(\Delta v_{h_i})^2 + (\Delta v_{h_\Omega})^2} \tag{2}$$

where

$$\Delta v_{h_i} = \frac{h}{r}\Delta i \tag{3}$$

$$\Delta v_{h_\Omega} = \frac{h}{r}\Delta\Omega \sin i \tag{4}$$

and where h is the orbit angular momentum and r the orbit equatorial radius. This normal impulse has an impact on i, Ω, and ω. Therefore, this effect on ω is compensated through another impulse.

The argument of periapsis and the mean anomaly are also corrected as a pair, but through two radial impulses. Those two impulses are to take place at apoapsis (Δv_{r_a}) and at periapsis (Δv_{r_p}). The magnitudes of the radial impulses are

$$\Delta v_{r_p} = -\frac{na}{4}\left[\frac{(1+e)^2}{\eta}(\Delta\omega + \Delta\Omega\cos i) + \Delta M\right] \tag{5}$$

$$\Delta v_{r_a} = -\frac{na}{4}\left[\frac{(1-e)^2}{\eta}(\Delta\omega + \Delta\Omega\cos i) + \Delta M\right] \tag{6}$$

where $n = \sqrt{\mu/a^3}$ is the orbit mean motion, μ the gravitational parameter, and $\eta = \sqrt{1-e^2}$. The remaining two tangential impulses are used to correct the orbit semimajor axis and the eccentricity. The two impulses are once again performed at periapsis and at apoapsis

$$\Delta v_{t_p} = \frac{na\eta}{4}\left(\frac{\Delta a}{a} + \frac{\Delta e}{1+e}\right) \tag{7}$$

$$\Delta v_{t_a} = \frac{na\eta}{4}\left(\frac{\Delta a}{a} - \frac{\Delta e}{1-e}\right) \tag{8}$$

Therefore, the near-optimal fuel cost of a small relative orbit element correction $\Delta\mathbf{e}$ can easily be estimated with the results of the impulsive feedback controller:

$$\Delta v = \sqrt{(\Delta v_{r_a})^2 + (\Delta v_{t_a})^2} + \sqrt{(\Delta v_{r_p})^2 + (\Delta v_{t_p})^2} + \sqrt{(\Delta v_{h_i})^2 + (\Delta v_{h_\Omega})^2} \tag{9}$$

This result is used to translate orbit element errors directly into fuel cost. It is used next to map these errors into the fuel-equivalent space.

III. Fuel-Equivalent Space

The relative orbit elements can be translated into six fuel-equivalent coordinates:

$$\delta\boldsymbol{V} = [\delta V_{t_p} \;\; \delta V_{t_a} \;\; \delta V_{h_i} \;\; \delta V_{h_\Omega} \;\; \delta V_{r_p} \;\; \delta V_{r_a}]^T \tag{10}$$

The δV_{t_p}, δV_{t_a}, δV_{r_p}, and δV_{r_a} coordinates represent the magnitude of the tangential [Eqs. (7) and (8)] and radial [Eqs. (5) and (6)] components of the apoapsis and periapsis impulses required to perform the Δa, Δe, $\Delta\Omega$, $\Delta\omega$, and ΔM corrections that would take the spacecraft from the origin to its current location. In turn, the δV_{h_i} and δV_{h_Ω} coordinates represent the impact on the magnitude of the normal impulse of Δi and $\Delta\Omega$ corrections [Eq. (2)]. In a formation flying context, the origin of the fuel-equivalent space would be the reference trajectory of the formation, or the leader. Thus, the coordinates of all the elements of the formation would be mapped in the same fuel-equivalent space with the same origin.

This linear mapping is thus performed through

$$\delta\boldsymbol{V} = S\delta\boldsymbol{e} \tag{11}$$

where the mapping matrix S is

$$S = \begin{bmatrix} \frac{n\eta}{4} & \frac{na\eta}{4(1+e)} & 0 & 0 & 0 & 0 \\ \frac{n\eta}{4} & -\frac{na\eta}{4(1-e)} & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{h}{r} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{h\sin i}{r} & 0 & 0 \\ 0 & 0 & 0 & -\frac{na}{4\eta}(1+e)^2\cos i & -\frac{na}{4\eta}(1-e)^2 & -\frac{na}{4} \\ 0 & 0 & 0 & -\frac{na}{4\eta}(1-e)^2\cos i & -\frac{na}{4\eta}(1-e)^2 & -\frac{na}{4} \end{bmatrix} \tag{12}$$

and $\delta\boldsymbol{e}$ is the relative orbit element vector with respect to the reference trajectory of the formation. This transformation translates the relative orbit elements into a six-dimensional space where the same displacement on any axis leads to an identical fuel cost. Thus, minimizing the distance in the fuel-equivalent space minimizes the fuel cost of a maneuver.

However, the true distance, in terms of fuel, between two points in the fuel-equivalent space is not the commonly used Euclidean norm. The use of the traditional Euclidean norm to compute distance leads to an underestimation of the fuel cost, because the Δv_a, Δv_p, and Δv_h corrections cannot be performed at the same time (they have to be performed at different locations of the orbit). Therefore, simultaneous displacements in the δV_{r_a}–δV_{t_a}, δV_{r_p}–δV_{t_p}, and the δV_{h_i}–δV_{h_Ω} planes are not allowed.

More rigorously, the fuel-equivalent distance d_{fe} between two points $\delta\boldsymbol{V}_1$ and $\delta\boldsymbol{V}_2$, in the six-dimensional fuel-equivalent space is defined as

$$d_{fe} = \sqrt{(\delta V_{r_{a_2}} - \delta V_{r_{a_1}})^2 + (\delta V_{t_{a_2}} - \delta V_{t_{a_1}})^2} + \sqrt{(\delta V_{r_{p_2}} - \delta V_{r_{p_1}})^2 + (\delta V_{t_{p_2}} - \delta V_{t_{p_1}})^2} + \sqrt{(\delta V_{h_{i_2}} - \delta V_{h_{i_1}})^2 + (\delta V_{h_{\Omega_2}} - \delta V_{h_{\Omega_1}})^2} \tag{13}$$

The distance d_{fe} provides an estimation of fuel cost of the optimal value of maneuvering from $\delta\boldsymbol{V}_1$ to $\delta\boldsymbol{V}_2$, with an accuracy similar to the results of the impulsive feedback controller.

This mapping thus translates the computation of minimal fuel cost into a geometric problem in the fuel-equivalent space. When desired formations can be described geometrically, systematic search of the optimal solution can be replaced by an analytical solution obtained by studying the geometry of the problem. This can be applied, for example, to the J_2-invariant orbits and the projected circular formation.

IV. Example of the J_2-Invariant Relative Orbits

This section shows how the fuel-equivalent space theory can be used to obtain a simple analytical solution to finding the closest (in terms of fuel) J_2-invariant relative orbit.

To a first-order approximation, the J_2-invariant conditions enforce the relative secular drift caused by J_2-perturbations of all elements to be zero, except for the argument of periapsis and the mean anomaly, for which the sum of the relative mean-element drift rates will be zero.

J_2-invariant orbits are defined by [13,14]

$$\delta a = -\frac{2Dae}{\eta}\delta e \tag{14}$$

$$\delta e = \frac{(1-e^2)\tan i}{4e}\delta i \tag{15}$$

where

$$D = \frac{J_2}{4L^4\eta^5}(4+3\eta)(1+5\cos^2 i) \tag{16}$$

$$L = \sqrt{a/R_e} \tag{17}$$

and where δa, δe, and δi are the relative semimajor axes, eccentricity and inclination of a *deputy* with respect to a *chief* (or a reference trajectory), and R_e is the planet's equatorial radius. This means that the admissible J_2-invariant relative orbits are defined by two linear constraints on the selection of δa, δe and δi [Eqs. (14) and (15)], which can be graphically represented as a straight line crossing the origin in the δa–δe–δi space. Because the mapping between the relative orbit element space and the fuel-equivalent space is linear, the J_2-invariant subset is also a straight line in the fuel-equivalent space. Because only δa, δe, and δi corrections will be required, only three of the six dimensions are relevant, and the mapping of this problem into the fuel-equivalent space takes a very simple form:

$$\begin{bmatrix} \delta V_{t_p} \\ \delta V_{t_a} \\ \delta V_{h_i} \\ 0 \\ 0 \\ 0 \end{bmatrix} = S \begin{bmatrix} \delta a \\ \delta e \\ \delta i \\ 0 \\ 0 \\ 0 \end{bmatrix} \tag{18}$$

so that the distance d_{fe} between two points $\delta \boldsymbol{V}_1$ and $\delta \boldsymbol{V}_2$ is determined by

$$d_{fe} = |\delta V_{t_{p_2}} - \delta V_{t_{p_1}}| + |\delta V_{t_{a_2}} - \delta V_{t_{a_1}}| + |\delta V_{h_{i_2}} - \delta V_{h_{i_1}}| \tag{19}$$

Therefore, finding the closest (in terms of fuel) J_2-invariant relative orbit from an initial relative orbit $\delta \boldsymbol{e}_0$ reduces to finding the location on a straight line (the J_2-invariant subset) that is the closest to a point (the initial spacecraft location), but with the distance defined as the sum of the absolute value of all the elements of the relative position vector.

The problem consists of finding the coordinates of the closest J_2-invariant relative orbit $\delta \boldsymbol{V}_2$ from the current coordinates $\delta \boldsymbol{V}_1$. Because the J_2-invariant subset is a straight line, $\delta \boldsymbol{V}_2$ needs to satisfy $\delta \boldsymbol{V}_2 = \delta \boldsymbol{V}_A + u(\delta \boldsymbol{V}_B - \delta \boldsymbol{V}_A)$ where $\delta \boldsymbol{V}_B$ and $\delta \boldsymbol{V}_A$ can be any two arbitrary but distinct J_2-invariant coordinates [satisfying Eqs. (14) and (15)], and u is a scalar.

If this constraint is enforced, the distance d_{fe} between $\delta \boldsymbol{V}_1$ and $\delta \boldsymbol{V}_2$ can be expressed as

$$\begin{aligned} d_{fe} = {} & |\delta V_{t_{p_1}} - \delta V_{t_{p_A}} - u(\delta V_{t_{p_B}} - \delta V_{t_{p_A}})| + |\delta V_{t_{a_1}} - \delta V_{t_{a_A}} \\ & - u(\delta V_{t_{a_B}} - \delta V_{t_{a_A}})| + |\delta V_{h_{i_1}} - \delta V_{h_{i_A}} - u(\delta V_{h_{i_B}} - \delta V_{h_{i_A}})| \end{aligned} \tag{20}$$

or more simply

$$d_{fe} = |d_{t_p}| + |d_{t_a}| + |d_{h_i}| \tag{21}$$

We seek the value of u that will minimize d_{fe}. Obviously, one of the ways of doing so is by studying the value of the derivative of d_{fe} with respect to u:

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}u} d_{fe} = {} & -\frac{|d_{t_p}|}{d_{t_p}}(\delta V_{t_{p_B}} - \delta V_{t_{p_A}}) - \frac{|d_{t_a}|}{d_{t_a}}(\delta V_{t_{a_B}} - \delta V_{t_{a_A}}) \\ & - \frac{|d_{h_i}|}{d_{h_i}}(\delta V_{h_{i_B}} - \delta V_{h_{i_A}}) \end{aligned} \tag{22}$$

The expression of the derivative of d_{fe} reveals that d_{fe} is a linear function of u between singularities. These singularities will happen when the derivative of d_{fe} is undefined, that is, when d_{t_p}, d_{t_a}, or d_{h_i} is zero. Thus, it can be graphically represented as four-line segments linked by three singularities.

Furthermore, because

$$\lim_{u\to\infty} d_{fe} = \infty \tag{23}$$

and

$$\lim_{u\to-\infty} d_{fe} = \infty \tag{24}$$

and because d_{fe} is linear between singularities, it can be shown that a minimum value for d_{fe} does exist and is inevitably found at one of the singularities. Indeed, the derivative of d_{fe} is a constant at every other location. Even if the derivative of d_{fe} is zero for a given line segment, the singularities found at the boundary of this line segment are still at the same distance as the complete line segment.

This leads to the convenient conclusion that the closest J_2-invariant orbit (in terms of fuel) will be located at $\delta V_{t_{p_2}} = \delta V_{t_{p_1}}$, $\delta V_{t_{a_2}} = \delta V_{t_{a_1}}$ or $\delta V_{h_{i_2}} = \delta V_{h_{i_1}}$. The combination of this result with the conditions for J_2-invariance leads to three potential maneuvers, one of which will be the most fuel-effective way to reach a J_2-invariant orbit.

The first case is $\delta V_{t_{p_2}} = \delta V_{t_{p_1}}$. This means that no tangential impulse is performed at periapsis to reach the J_2-invariant orbit. This yields the condition

$$\Delta v_{t_p} = \frac{na\eta}{4}\left(\frac{\Delta a}{a} + \frac{\Delta e}{1+e}\right) = 0 \tag{25}$$

where the corrections Δa and Δe are the maneuvers required to reach the J_2-invariance conditions δa_{inv}, δe_{inv} and δi_{inv} from the current coordinates δa_0, δe_0 and δi_0

$$\Delta a = \delta a_0 - \delta a_{\mathrm{inv}} \tag{26}$$

$$\Delta e = \delta e_0 - \delta e_{\mathrm{inv}} \tag{27}$$

$$\Delta i = \delta i_0 - \delta i_{\mathrm{inv}} \tag{28}$$

However, the J_2-invariance constraints also have to be enforced:

$$\delta a_{\mathrm{inv}} = -\frac{2Dae}{\eta}\delta e_{\mathrm{inv}} \tag{29}$$

$$\delta e_{\mathrm{inv}} = -\frac{(1-e^2)\tan i}{4e}\delta i_{\mathrm{inv}} \tag{30}$$

Through algebraic manipulations, Eqs. (25–30) can be combined to yield the set of orbit element corrections Δa_1, Δe_1, and Δi_1 that will lead to J_2-invariant orbits at the first singularity:

$$\Delta a_1 = \frac{-\eta\delta a_0 + 2Dae\delta e_0}{\eta - 2De(1+e)} \tag{31}$$

$$\Delta e_1 = \frac{\eta(1+e)\delta a_0 - 2Dae(1+e)\delta e_0}{a\eta - 2Dae(1+e)} \tag{32}$$

$$\Delta i_1 = \frac{4e(1+e)\delta a_0 - 4ae\delta e_0}{a\eta[\eta - 2De(1+e)]\tan i} - \delta i_0 \tag{33}$$

The second condition requires that $\delta V_{t_{a_2}} = \delta V_{t_{a_1}}$, i.e., no tangential impulse at apoapsis is performed. This condition implies that

$$\Delta v_{t_a} = \frac{na\eta}{4}\left(\frac{\Delta a}{a} - \frac{\Delta e}{1-e}\right) = 0 \tag{34}$$

Combining Eq. (34) with Eqs. (26–30) leads to a second set of corrections, Δa_2, Δe_2, and Δi_2, that could be the most fuel-efficient way to reach J_2-invariance:

$$\Delta a_2 = \frac{-\eta\delta a_0 - 2Dae\delta e_0}{\eta + 2De(1-e)} \tag{35}$$

$$\Delta e_2 = \frac{-\eta(1-e)\delta a_0 - 2Dae(1-e)\delta e_0}{a\eta + 2Dae(1-e)} \tag{36}$$

$$\Delta i_2 = \frac{-4e(1-e)\delta a_0 + 4ae\delta e_0}{a\eta[\eta + 2De(1-e)]\tan i} - \delta i_0 \tag{37}$$

Finally, the third and last potential set of corrections is at $\delta V_{h_{i_2}} = \delta V_{h_{i_1}}$, i.e., no inclination correction is performed. Forcing $\Delta v_{h_i} = 0$ imposes $\Delta i = 0$ and yields a third set of corrections to reach J_2-invariance conditions:

$$\Delta a_3 = -\delta a_0 - \frac{Da\tan i}{2}\delta i_0 \tag{38}$$

$$\Delta e_3 = -\delta e_0 + \frac{\eta^2\tan i}{4e}\delta i_0 \tag{39}$$

$$\Delta i_3 = 0 \tag{40}$$

Therefore, the most fuel-efficient way of reaching a J_2-invariant orbit is the ith set of conditions; out of the three, that will be the least expensive in terms of fuel. This fuel cost can easily be computed by mapping those corrections in the fuel-equivalent space and by computing the distance of the fuel-equivalent coordinates with respect to origin:

$$\delta \boldsymbol{V}_i = S\begin{bmatrix} \Delta a_i \\ \Delta e_i \\ \Delta i_i \\ 0 \\ 0 \\ 0 \end{bmatrix} \tag{41}$$

with

$$d_{fe_i} = |\delta V_{t_{p_i}}| + |\delta V_{t_{a_i}}| + |\delta V_{h_{i_i}}| \tag{42}$$

Results can be further simplified by looking at the order of magnitude of the difference between the two first sets of corrections. The differences in terms of required corrections for the first two conditions are

$$\Delta a_1 - \Delta a_2 = \frac{-4\eta De\delta a_0 - 8D^2ae^2\delta e_0}{\eta^2 - 4\eta De^2 - 4D^2e^2 + 4D^2e^4} \tag{43}$$

$$\Delta e_1 - \Delta e_2 = \frac{2\eta^2\delta a_0 - (8D^2ae^2 + 4Dae^2\eta - 8D^2ae^4)\delta e_0}{a\eta^2 - 4a\eta De^2 - 4D^2ae^2 + 4aD^2e^4} \tag{44}$$

$$\Delta i_1 - \Delta i_2 = \frac{(-16De^2 - 8e^2\eta + 16De^4)\delta a_0 + 16aDe^2\delta e_0}{a\eta\tan i(\eta^2 - 4\eta De^2 - 4D^2e^2 + 4D^2e^4)} \tag{45}$$

For typical 1 km size low-Earth orbit (LEO) formations, one can conservatively assume that

$$\mathcal{O}(a) = 10^7\ \text{m} \tag{46}$$

$$\mathcal{O}(e) = 10^{-2} \tag{47}$$

Table 1 Chief initial orbit elements

	$\boldsymbol{e}_0$
a_0	$1.1R_e$
e_0	0.05
i_0	$\pi/4$
Ω_0	0
ω_0	0
M_0	0

Table 2 Deputy initial orbit elements offset

	$\delta\boldsymbol{e}_0$
δa_0	0
δe_0	+0.0001
δi_0	+0.0001
$\delta\Omega_0$	−0.0001
$\delta\omega_0$	−0.0001
δM_0	+0.0001

$$\mathcal{O}(\delta a_0) = 10^2\ \text{m} \tag{48}$$

$$\mathcal{O}(\delta e_0) = 10^{-3} \tag{49}$$

$$\mathcal{O}(\delta i_0) = 10^{-3} \tag{50}$$

$$\mathcal{O}(D) = 10^{-2} \tag{51}$$

Under these assumptions, and assuming the orbit is not near-equatorial [$\mathcal{O}(\tan i) \geq 1$], the order of magnitude of the required corrections differences are

$$\mathcal{O}(\Delta a_1 - \Delta a_2) = 10^{-2}\ \text{m} \tag{52}$$

$$\mathcal{O}(\Delta e_1 - \Delta e_2) = 10^{-5} \tag{53}$$

$$\mathcal{O}(\Delta i_1 - \Delta i_2) = 10^{-8} \tag{54}$$

At worst, these differences lead to an impact on the required fuel in the order of a few cm/s. Practically speaking, condition 1 and condition 2 lead to orbit element corrections that cannot be distinguished one from each other. Therefore, for inclined 1 km size LEO formations, the fuel cost needs to be computed for only two points. The most fuel-efficient way to reach a J_2-invariant orbit is the first set of corrections, Δa_1, Δe_1, and Δi_1 (which are similar to Δa_2, Δe_2, and Δi_2), or the third set of corrections, Δa_3, Δe_3, and Δi_3, that requires no inclination corrections.

A numerical example is given next to illustrate the results. The reference orbit is described in Table 1. The deputy is given a small orbit element offset $\delta\boldsymbol{e}_0$ as shown in Table 2. The problem is to find the closest J_2-invariant location starting from the initial location of the deputy.

The results of a systematic search are given in Fig. 1. Equations (31–33) and (38–40) avoid the need for a systematic search, as they predict analytically the location of the singularities in the distance function where the minimum will be found. As expected, both condition 1 and condition 2 lead to practically the same correction. Both singularities cannot be reasonably distinguished. They both are located at the no tangential burns singularity location. Therefore, the most fuel-efficient way to reach a J_2-invariant relative

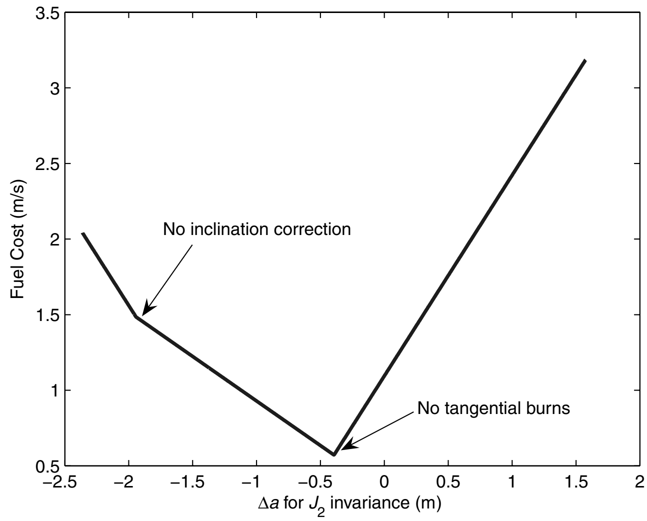

Fig. 1 Fuel cost as a function of Δa required for J_2 invariance.

orbit from the initial conditions of Tables 1 and 2 is to perform the corrections $\Delta a = -0.38$ m, $\Delta e = -5.8 \times 10^{-8}$, and $\Delta i = -8.0 \times 10^{-5}$ as given by Eqs. (31–33).

V. Example of the Projected Circular Formation

The fuel-equivalent-space theory can also be applied to geometrically defined formations such as the PCF. This theory can be applied to quickly identify the closest (in terms of fuel) position on a PCF without the need for a systematic search.

The PCF is a formation for which all members of the formation are at the same distance from the center of the formation in the normal-tangential plane (Fig. 2). In other words, as seen from Earth, all members are distributed on a circle. This could have several application for Earth observation.

In Hill coordinates, the projected circular formation is constrained by

$$\rho^2 = \sqrt{y^2 + z^2} \tag{55}$$

where ρ is the radius of the PCF. All the admissible sets of Hill coordinates for a given ρ can be obtained by sweeping the circular formation angular position β:

$$x(\beta) = -\frac{\rho}{2}\cos\beta \tag{56}$$

$$y(\beta) = \rho\sin\beta \tag{57}$$

$$z(\beta) = \rho\cos\beta \tag{58}$$

$$V_x(\beta) = \frac{\rho n}{2}\sin\beta \tag{59}$$

$$V_y(\beta) = \rho n\cos\beta \tag{60}$$

$$V_z(\beta) = \rho n\sin\beta \tag{61}$$

With an accurate and simple relative motion model, the required set of relative orbit elements $\delta \boldsymbol{e}_{\mathrm{PCF}}(\beta)$ to reach the corresponding set of Hill coordinates at a desired orbit location can be obtained:

$$\delta \boldsymbol{e}_{\mathrm{PCF}}(\beta) = \Phi^{-1}\delta X(\beta) \tag{62}$$

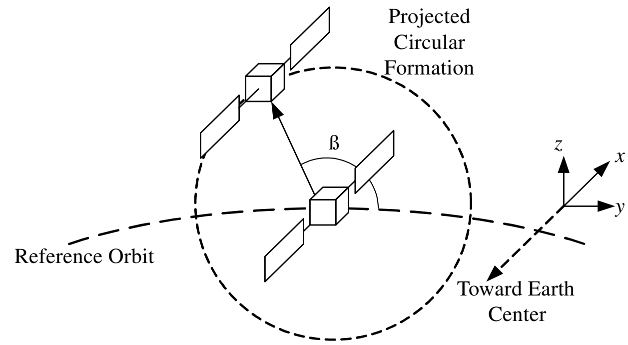

Fig. 2 Projected circular formation in Hill coordinates.

where $\delta X(\beta) = [x \;\; y \;\; z \;\; V_x \;\; V_y \;\; V_z]^T$. The matrix Φ can be a simple linearization of the mapping between relative orbit elements and Hill coordinates, if the formation is desired at the current location, or it can be a state transition matrix if the formation is required at another point farther on the orbit. The only requirement is that the matrix Φ relates current relative orbit elements into Hill coordinates at the point where the projected circular formation is desired.

We seek the angle $\beta_{\min}$ for which the fuel cost of maneuvering from the current relative orbit elements $\delta \boldsymbol{e}_0$ to $\delta \boldsymbol{e}_{\mathrm{PCF}}(\beta)$ is minimized. Once the corresponding coordinates $\delta \boldsymbol{V}_1$ and $\delta \boldsymbol{V}_{\mathrm{PCF}}(\beta)$ have been identified in the fuel-equivalent space, the most fuel-efficient way to reach a projected circular formation starting from $\delta \boldsymbol{e}_0$ is the set of elements $\delta \boldsymbol{e}_{\mathrm{PCF}}(\beta)$ for which the distance between $\delta \boldsymbol{V}_1$ and $\delta \boldsymbol{V}_{\mathrm{PCF}}(\beta)$ is minimized.

The traditional way of solving this problem is to systematically compute the fuel cost to reach each location on the formation for the whole range of β, i.e., between 0 and 2π, such as done by Mueller [2]. However, the translation of the problem into the fuel-equivalent space leads to simple geometric relationships that can provide the closest location on a projected circular formation without the need for a systematic search. The problem is reduced to finding the minimal distance (and location of the minimal distance) between a point and an ellipse in a 6-D space.

Theoretically, distances in the fuel-equivalent space are measured in terms of fuel distance d_{fe}:

$$d_{fe} = \sqrt{\delta V_{t_p}^2 + \delta V_{r_p}^2} + \sqrt{\delta V_{t_a}^2 + \delta V_{r_a}^2} + \sqrt{\delta V_{h_i}^2 + \delta V_{h_\Omega}^2} \tag{63}$$

instead of the more common Euclidean distance d_{Euc}:

$$d_{\mathrm{Euc}} = \sqrt{\delta V_{t_p}^2 + \delta V_{r_p}^2 + \delta V_{t_a}^2 + \delta V_{r_a}^2 + \delta V_{h_i}^2 + \delta V_{h_\Omega}^2} \tag{64}$$

However, to be able to use the Euclidean distance provides the advantage that its derivative with respect to β is always continuous, which is not the case with the fuel distance.

Using the Euclidean distance systematically underestimates the fuel cost, as it assumes that all the impulses can be performed at the same time. However, there will always exist a constant K for which

$$d_{fe} = K d_{\mathrm{Euc}} \tag{65}$$

where

$$1 < K \leq \sqrt{3} \tag{66}$$

K is equal to one if one of the three impulses (δV_p, δV_a, or δV_h) is infinitely larger than the other two. In this case, both distances' computations become identical. On the other hand, $K = \sqrt{3}$ if all three impulses are identical. It is very unlikely to find both extremes in one single PCF. It is even more unlikely for this to happen for a small variation of β. Therefore, it is reasonable to assume that K is approximately constant for a given PCF. This essentially means that both distance functions have the same shape, so that the β for minimum distance is located at the same place for the two functions.

Therefore, finding the β for minimal d_{Euc} is a way of finding the β for minimum d_{fe}.

If instead one seeks to minimize the Euclidean distance, this can be done by minimizing the function D:

$$D = \tfrac{1}{2}(\delta \boldsymbol{V}^T \delta \boldsymbol{V}) \tag{67}$$

where

$$\delta \boldsymbol{V} = S[\delta e_0 - \Phi^{-1} \delta X(\beta)] \tag{68}$$

The distance D has the following first and second derivative expressions:

$$\frac{\mathrm{d}D}{\mathrm{d}\beta} = \delta \boldsymbol{V}^T S \Phi^{-1} \frac{\mathrm{d}}{\mathrm{d}\beta} \delta X(\beta) \tag{69}$$

$$\frac{\mathrm{d}^2 D}{\mathrm{d}\beta^2} = (S\delta e_0)^T S \Phi^{-1} \delta X(\beta) + \left[S\Phi^{-1} \frac{\mathrm{d}}{\mathrm{d}\beta} \delta X(\beta)\right]\left[S\Phi^{-1} \frac{\mathrm{d}}{\mathrm{d}\beta} \delta X(\beta)\right] - [S\Phi^{-1}\delta X(\beta)]^T [S\Phi^{-1}\delta X(\beta)] \tag{70}$$

The fully expanded D is of the form:

$$D = A_0 + A_1 \cos(\beta) + B_1 \sin(\beta) + A_2 \cos(2\beta) + B_2 \sin(2\beta) \tag{71}$$

The coefficients of the distance function could be expressed in terms of $\delta \boldsymbol{e}_0$, n, ρ and the components of Φ^{-1}. However, the development of this expression would be tedious, and the identification of the minimum of the function would not be straightforward. Because of the sinusoidal nature of the function, and because first and second derivatives are known (and continuous), it is much more efficient to locate $\beta_{\min}$ iteratively starting from an educated guess.

The location of the minimum of the function, where $\mathrm{d}D/\mathrm{d}\beta = 0$ and $\mathrm{d}^2 D/\mathrm{d}\beta^2 > 0$ can be obtained very quickly with a Newton–Raphson iteration of the form

$$\beta[k+1] = -\frac{\mathrm{d}D/\mathrm{d}\beta}{\mathrm{d}^2 D/\mathrm{d}\beta^2} + \beta[k] \tag{72}$$

If the first guess is the angular position β with the smallest distance out of a sufficiently large number (typically six) of regularly-spaced initial guesses between 0 and 2π, the algorithm usually converges to the function minimum within two or three iterations.

This is illustrated next by a numerical example. The chief orbit element vector $\boldsymbol{e}_0$ (the center of the formation) is set to a slightly elliptical 45 deg inclined low-Earth orbit, as described in Table 1. The deputy is given a small orbit element offset $\delta \boldsymbol{e}_0$ as shown in Table 2. The problem consists in finding the angular position β on the projected circular formation that will the be the least expensive to reach within a time frame of one orbit.

The fuel distance and the Euclidean distance for the whole range of β is shown in Fig. 3. As can be seen on this figure, the use of the Euclidean norm systematically underestimates the fuel cost of the maneuver. However, both types of distance correctly locate the β for minimum effort at $\beta = 1.01$. The optimal fuel cost of Fig. 3 is the fuel cost of the maneuver, computed using optimal control theory [6] going from the initial position to the desired PCF location one orbit later at time t_f and minimizing the cost function J:

$$J = \int_0^{t_f} \left(\frac{1}{2} \boldsymbol{u}^T \boldsymbol{u}\right) \mathrm{d}t \tag{73}$$

where $\boldsymbol{u}$ is the control vector. As opposed to the impulsive feedback controller theory, the optimal control theory assumes continuous firing of the thrusters and seeks to minimize a quadratic control effort. However, as shown in Fig. 3, both functions have their minimum located at the same β. This shows that the fuel-equivalent space theory can be used even with continuous thrusters, assuming the optimal fuel cost difference between using continuous thruster firing and using impulsive thruster firing remains small for a given formation.

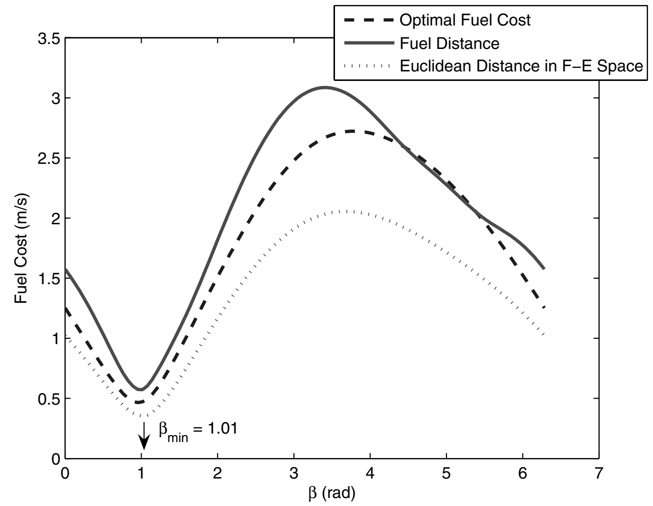

Fig. 3 Fuel cost as a function of PCF angular position.

The same process can be applied to any type of formation that can be described geometrically. The use of geometric relations avoids the need for systematic search and transforms the problem into minimizing the distance between a point (the current location of the spacecraft) and a geometric shape (that represents the desired formation) at the cost of a few Newton–Raphson-type iterations.

VI. Conclusions

A new tool, the fuel-equivalent space, useful to rapidly compute the most fuel-efficient way to reach a desired formation, has been presented. This theory is based on analytical and simple relations and is therefore well suited for autonomous on-board application.

The method maps the required relative orbit elements corrections into a six-dimensional fuel-equivalent space in which a similar displacement on each of the axes requires the same amount of fuel. Therefore, the fuel cost of a given maneuver is minimized if the distance is minimized in the fuel-equivalent space.

Two examples of application of the fuel-equivalent space theory have been presented. The first one is the J_2-invariant relative orbits. In this case, finding the closest (in terms of fuel) J_2-invariant relative orbit is reduced to computing the fuel cost of only two possible maneuvers, one of which is to perform no inclination maneuver. The second application is the PCF. Once mapped into the fuel-equivalent space, all the possible circular formation locations for a given formation radius form an ellipse in the six-dimensional fuel-equivalent space. In this case, the problem is reduced to finding the minimum distance between a point and an ellipse. It has been shown that the Euclidean norm can effectively be used to locate the minimum of the fuel distance function. This distance has simple first and second derivatives, and its global minimum can be identified with few iterations.

This theory can therefore be applied to any formation that can be geometrically defined in the relative orbit element space. Finding the most fuel-efficient way to reach the formation reduces to finding the minimum distance between a point and a geometric shape. If the formation has a geometrically simple shape (as is the case with the J_2-invariant orbit and the PCF), simple analytical relations can be established, and the most fuel-efficient maneuver can be identified analytically or with very few simple iterations.

References

[1] Tillerson, M., Inalhan, G., and How, J. P., "Co-Ordination and Control of Distributed Spacecraft Systems Using Convex Optimization Techniques," *International Journal of Robust and Nonlinear Control*, Vol. 12, No. 2–3, 2002, pp. 207–242. doi:10.1002/rnc.683

[2] Mueller, J. B., "A Multiple-Team Organization for Decentralized Guidance and Control of Formation Flying Spacecraft," *AIAA Intelligent Systems Conference*, AIAA, Reston, VA, Sept. 2004.

[3] Yang, G., Yang, Q., Kapila, V., Palmer, D., and Vaidyanathan, R., "Fuel Optimal Manoeuvres for Multiple Spacecraft Formation Reconfiguration Using Multi-Agent Optimization," *International Journal of Robust and Nonlinear Control*, Vol. 12, Nos. 2–3, 2002, pp. 243–283.
doi:10.1002/rnc.684

[4] Hoskins, A. B., and Atkins, E. M., "Spacecraft Formation Optimization with a Multi-Impulse Design," AIAA Paper 2005-5835, Aug. 2005.

[5] Avanzini, G., Biamonti, D., and Minisci, E. A., "Minimum-Fuel/ Minimum-Time Maneuvers of Formation Flying Satellites," AAS Paper 03-654, Aug. 2003.

[6] Bryson, A., and Ho, Y.-C., *Applied Optimal Control*, Hemisphere Publishing Corporation, Washington, D.C., 1975.

[7] Wang, P. K. C., and Hadaegh, F. Y., "Minimum-Fuel Formation Reconfiguration of Multiple Free-Flying Spacecraft," *Journal of the Astronautical Sciences*, Vol. 47, Nos. 1–2, 1999, pp. 77–102.

[8] Sabol, C., Burns, R., and McLaughlin, C. A., "Satellite Formation Flying Design and Evolution," *Journal of Spacecraft and Rockets*, Vol. 38, No. 2, March–April 2001, pp. 270–278.

[9] Mishne, D., "Maintaining Periodic Relative Trajectories of Satellite Formation by Using Power-Limited Thrusters," AAS Paper 03-656, Aug. 2003.

[10] Vaddi, S. S., Alfriend, K. T., Vadali, S. R., and Sengupta, P., "Formation Establishment and Reconfiguration Using Impulsive Control," *Journal of Guidance, Control, and Dynamics*, Vol. 28, No. 2, March–April 2005, pp. 262–268.

[11] Gurfil, P., "Relative Motion Between Elliptic Orbits: Generalized Boundedness Conditions and Optimal Formationkeeping," *Journal of Guidance, Control, and Dynamics*, Vol. 28, No. 4, July–Aug. 2005, pp. 761–767.

[12] Schaub, H., and Alfriend, K. T., "Impulsive Feedback Control to Establish Specific Mean Orbit Elements of Spacecraft Formations," *Journal of Guidance, Control, and Dynamics*, Vol. 24, No. 4, July–Aug. 2001, pp. 739–745.

[13] Schaub, H., and Junkins, J. L., *Analytical Mechanics of Space Systems*, AIAA Education Series, AIAA, Reston, VA, 2003.

[14] Schaub, H., "Relative Orbit Geometry Through Classical Orbit Element Differences," *Journal of Guidance, Control, and Dynamics*, Vol. 27, No. 5, Sept.–Oct. 2004, pp. 839–848.